Asymptotic behavior of the stock vector in a mixed push-pull manpower model

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Abstract

In this paper, the asymptotic behavior of the time-homogeneous mixed push-pull manpower model is studied under the assumption that the desired stock vector and the recruitment policy are fixed over time. In the mixed push-pull manpower model, the internal mobility of a personnel system can be regulated by both pull and push transitions. Based on those characteristics, we express and examine the dynamics of the personnel system by formulating the mixed push-pull manpower model by means of particular transition matrices, which we demonstrate to have interesting properties. We show that under certain conditions the stock vector converges. An explicit analytical form for this limiting personnel stock vector is found.

Keywords: Manpower planning; stochastic models; push models; pull models; asymptotic behavior.

1. Introduction

Manpower planning provides long term organizational decision support regarding recruitments, redundancies and internal staff mobility. Based on aggregate analyses, various mathematical models for manpower planning have been developed to describe, predict and control the future personnel availability in the different states of a manpower system [1,2]. The total population is therefore classified into homogeneous groups of employees, according to various employee characteristics, like grade, skills, knowledge and abilities [3-5]. The evolution of personnel availability fully depends on the recruitment and wastage in each group and on the personnel mobility among those groups. Hence, to study this personnel dynamics, organizations are modeled as systems of stocks and flows.

Although recently alternative approaches have been used (e.g. [1,6-8]), most mathematical models for manpower planning are founded on Markov theory [9-11]. Those so-called push models assume that all employees within a homogenous group are subject to the same internal
transition and wastage probabilities. In general, three types of push models can be distinguished. First, time-homogeneous Markov models are suitable for organizations in which the personnel flows among groups can be defined as time independent transition probabilities (e.g. [12,13]). It is well known that the flows follow a multinomial distribution, for which the parameters, i.e. the transition probabilities, need to be estimated [14]. Second, non homogeneous Markov models relax this assumption and include time dependent transition probabilities as model parameters (e.g. [15-18]). Finally, semi-Markov models relax the assumption of homogeneous subgroups, by considering conditional transition probabilities, depending on the individual’s duration of group membership (e.g. [19-21]).

Once hired, employees grow in terms of skills, knowledge and abilities. Hence, time-discrete push models are very suitable to model personnel flows, since in each time interval a certain number of employees is expected to make a transition among states determined by these personnel characteristics. While considerable research has been devoted to push models in manpower planning, less attention has been paid to the so-called pull models. Based on Renewal theory, in pull models, a transition into a state is only possible if there are vacancies to be filled [2]. The number of vacancies is thereby assumed to follow a binomial distribution, based on the wastage probability in the receiving state.

In practice, a mix of push and pull transitions might occur in the same personnel system. Therefore, Georgiou and Tsantas [22] introduced a hybrid model, which was further developed by Dimitriou and Tsantas [23,24]. This approach allows modeling push as well as pull flows within the same personnel system. Besides push flows between active classes, a preparation class is introduced from which individuals can be pulled towards the active classes in case vacancies arise in the active states. However, this approach is restricted to an embedded Markov model, assuming known total system size, implying that the desired total number of individuals in the system is known and the vacancies are determined at an aggregated level. In certain organizations however, the desired number of employees and the number of vacancies are specified at group level. De Feyter [25] therefore introduced the mixed push-pull manpower model, which allows modeling push as well as pull flows between all states in the personnel system, by considering vacancies in the individual states. Accordingly, the mixed push-pull
The manpower model is a generalization of the traditional push and pull models. The model is very suitable for organizations in which the states are defined among others by the grades in the personnel hierarchy. This way, the mixed push-pull approach allows modeling personnel transitions that take place because of vacancies arise at higher levels in the personnel system (pull flows) or employees attain skills, knowledge and abilities that are necessary for grades at a higher level in the hierarchical system (push flows). Hence, the flows can be determined by push and pull mechanisms. Consequently, in comparison to traditional push and pull models, in a mixed push-pull model, it is harder to examine the future personnel dynamics. For a state in the personnel system, depending on the vacancies in other states in the system, the mechanism (push, pull or push-pull) that determines the flows may differ over time. In some time intervals, when there are vacancies at higher levels, both push and pull promotions would occur, while in other time intervals, when there are no open positions to be filled in higher levels, people would only be promoted by the push mechanism. Furthermore, the intervening transition mechanism can be different for the distinct states of the manpower system.

A central issue in manpower planning research is the attainability and maintainability problem [23,26-30]. For a manpower system, one of the objectives is to control the evolution of the personnel system in order to attain or maintain a desirable stock in the future. Another important concern in manpower planning is the direction in which the personnel structure is changing [24,31]. Therefore, research in manpower planning is very interested in the asymptotic behavior of personnel systems. For the mixed push-pull manpower model, De Feyter [25] discussed the maintainability and attainability problem under control by recruitment. Furthermore, some preliminary results were found on the asymptotic behavior of the time-homogeneous mixed push-pull model, under the assumption that the workforce demand and the recruitment policy are fixed over time. For organizations in which the transition mechanism is determined by both pull and push flows in all states of the personnel system and during all time intervals, it is shown that under certain conditions the stock vector converges. The numerical illustrations however reveal that these are sufficient but not necessary conditions for convergence towards a limiting stock vector in mixed push-pull personnel systems.
In the present paper, we further elaborate on the asymptotic behavior of the mixed push-pull manpower model. First, we extend the investigation by relaxing the restriction that for each state push as well as pull flows are involved. Second, we allow the intervening transition mechanism to differ between the states of the manpower system.

This paper proceeds with a notion of the mixed push-pull model in the next section. A set of difference equations is presented, as well as its specific notations and properties, what allows describing the evolution of the stocks in a mixed push-pull approach. In Section 3, by introducing specific transition matrices $M(v_i(t))$, we propose a reformulation of the mixed push-pull model. As is shown in Section 4, these transition matrices have some interesting properties, which are very useful to examine the asymptotic behavior of the mixed push-pull model in Section 5. Finally, Section 6 provides an illustration of the theoretical results of the preceding sections.

2. The mixed push-pull model

This paragraph provides the mixed push-pull as presented in De Feyter [25]. For a personnel system divided into $k$ homogeneous groups, the stocks at time $t \in \mathbb{N}$ are denoted by the row vector $n(t) = (n_1(t), \ldots, n_k(t))$ with $n_i(t)$ being the expected number of employees in homogeneous group $i$ at time $t$. Let $W(t)$ be a $k \times k$ diagonal matrix with $W_{ii}(t) = w_i(t)$ the wastage probability in group $i$ in time interval $[t-1,t)$. Further, we define the row vector $V(t) = (V_1(t), \ldots, V_k(t))$ with $V_i(t)$ being the number of vacancies in group $i$ to be filled in time interval $[t-1,t)$. The mixed push-pull model assumes the vacancies to be $V(t) = Max(0, n^*(t) - n(t-1)[I - W(t)])$ with $n^*(t) = (n^*_i(t))$ being the $1 \times k$ row vector with $n^*_i(t)$ the desired number of employees in group $i$ at time $t$. The Max-operator is hereby defined as $Max(A, B) = (Max(A_1, B_1), \ldots, Max(A_k, B_k))$.

The internal dynamics of the personnel system in the mixed push-pull model is firstly regulated by pull transitions to fill the vacancies, governed by the $k \times k$ matrix $S(t) = (s_{ij}(t))$ with $s_{ij}(t)$
being the probability in time interval \([t-1,t)\) for a vacancy in group \(i\) to be filled by an employee from group \(j\).

Remark that for every group \(j\) and at each time \(t\) the stock should be sufficiently large to capture all pull transitions from that group, i.e.

\[
\forall j, t: \quad n_j(t) - n_j(t-1) \left[1 - w_j(t)\right] \geq \sum_i V_i(t) s_{ij}(t)
\] (1)

Note that \(S(t)\) is not necessarily a row stochastic matrix. Indeed, the mixed push-pull model assumes that not all vacancies need to be filled by internal transitions, but can also be captured by pull recruitments. Subsequently, the internal mobility of the mixed push-pull model is supplemented by push transitions. The push flows are characterized by the transition matrix \(P(t) = \{p_{ij}(t)\}\), and by the row stochastic transition matrix \(Q(t) = \{q_{ij}(t)\}\). Its elements \(q_{ij}(t)\) represent the transition probabilities for employees in group \(i\) at time \(t-1\) to be in group \(j\) at time \(t\), under the condition that they do not leave the system nor make a pull transition during time interval \([t-1,t)\):

\[
[I - W(t)]Q(t) = P(t)
\] (2)

The total number of recruitments additional to the pull recruitments at time \(t\) is \(R(t)\) and \(r_i(t)\) is the proportion of \(R(t)\) assigned to group \(i\). The \(1 \times k\) row vector \(r(t) = \{r_i(t)\}\) is the recruitment vector.

The expected number of personnel in every homogeneous group can be calculated by a system of difference equations:

\[
n(t) = V(t) + \left\{n(t-1) \left[I - W(t)\right] - V(t) S(t)\right\} Q(t) + R(t) r(t)
\] (3)

\[
V(t) = \text{Max} \left\{0, n^*(t) - n(t-1) [I - W(t)]\right\}
\] (4)

\[
w_i(t) = 1 - \sum_{j=1}^{k} p_{ij}(t) \quad \text{for} \quad i = 1, \ldots, k
\] (5)
3. A reformulation of the mixed push-pull model

According to (4), there are vacancies in group \(i\) to be filled in case
\[
V_i(t) > 0 \quad \Leftrightarrow \quad n_i^*(t) - n_i(t-1)[1 - w_i(t)] > 0
\]
By introducing for all \(i \in \{1, \ldots, k\}\) the matrix \(I_i\) satisfying
\[
(I_i)_{ji} = 1 \quad \text{for} \quad (j, l) = (i, i) \quad \text{and} \quad (I_i)_{ji} = 0 \quad \text{for} \quad (j, l) \neq (i, i)
\]
and by setting
\[
\nu_i(t) = \{i \mid V_i(t) > 0\}
\]
\[
A(t) = I - S(t)Q(t)
\]
\[
M(\nu_i(t)) = [I - W(t)] \cdot \left( Q(t) - \sum_{i \in \nu_i(t)} I_i \cdot A(t) \right)
\]
(3) and (4) can be written in terms of the matrices \(I_i, A(t)\) and \(M(\nu_i(t))\):

**Theorem 1.**

\[
V(t) = \left[ n^*(t) - n(t-1)(I - W(t)) \right] \cdot \sum_{i \in \nu_i(t)} I_i
\]
(6)
\[
n(t) = n(t-1)M(\nu_i(t)) + n^*(t) \cdot \sum_{i \in \nu_i(t)} I_i \cdot A(t) + R(t)r(t)
\]
(7)

**Proof.**

On the one hand, for \(i \in \nu_i(t)\) holds that \(V_i(t) > 0\), what is equivalent with
\[
n_i^*(t) - n_i(t-1)(1 - w_i) > 0.
\]
Moreover, by definition of the matrices \(I_i\), the \(i\)-th coordinate of
\[
\left[ n^*(t) - n(t-1)(I - W(t)) \right] \cdot \sum_{i \in \nu_i(t)} I_i
\]
equals \[
n_i^*(t) - n_i(t-1)(1 - w_i).
\]
On the other hand, for \(i \notin \nu_i(t)\) holds that the \(i\)-th coordinate of
\[
Max\left[0, n_i^*(t) - n(t-1)(I - W(t))\right]
\]
as well as of \[
\left[ n_i^*(t) - n(t-1)(I - W(t)) \right] \cdot \sum_{i \in \nu_i(t)} I_i
\]
equals zero.
Consequently
\[
Max\left[0, n_i^*(t) - n(t-1)(I - W(t))\right] = \left[ n_i^*(t) - n(t-1)(I - W(t)) \right] \cdot \sum_{i \in \nu_i(t)} I_i
\]
which proves (6).
According to (3) the evolution of the stock vector can be expressed as:

\[ n(t) = V(t) + \left\{ n(t-1) \left[ I - W(t) \right] - V(t) S(t) \right\} Q(t) + R(t) r(t) \]

\[ = V(t) \left[ I - S(t) Q(t) \right] + n(t-1) \left[ I - W(t) \right] \]

By (6) and by the definition of the matrices \( A(t) \) and \( M(\nu_v(t)) \) this expression can be rewritten as:

\[ n(t) = n^*(t) - n(t-1) \left[ I - W(t) \right] \sum_{i \in \nu_v(t)} I_{ii} \left[ I - S(t) Q(t) \right] + n(t-1) \left[ I - W(t) \right] \]

\[ = n(t-1) \left[ I - W(t) \right] \left\{ Q(t) - \sum_{i \in \nu_v(t)} I_{ii} \left[ I - S(t) Q(t) \right] \right\} + n^*(t) \sum_{i \in \nu_v(t)} I_{ii} \]

\[ = n(t-1) \left[ I - W(t) \right] \left\{ Q(t) - \sum_{i \in \nu_v(t)} I_{ii} A(t) \right\} + n^*(t) \sum_{i \in \nu_v(t)} I_{ii} A(t) + R(t) r(t) \]

\[ = n(t-1) M(\nu_v(t)) + n^*(t) \sum_{i \in \nu_v(t)} I_{ii} A(t) + R(t) r(t) \]

Which proves (7) and the Theorem.

The convergence properties of the stock vector \( n(t) \) will be discussed in terms of the matrices \( M(\nu_v(t)) \). Therefore in first instance these matrices are examined in the next paragraph.

4. Properties of the matrices \( M(\nu_v(t)) \)

Since \( M(\nu_v(t)) = \left[ I - W(t) \right] \left( Q(t) - \sum_{i \in \nu_v(t)} I_{ii} A(t) \right) \) and \( A(t) = I - S(t) Q(t) \), it holds that:

\[ M(\nu_v(t)) = \left[ I - W(t) \right] \left( \left[ I + \sum_{i \in \nu_v(t)} I_{ii} S(t) \right] Q(t) - \sum_{i \in \nu_v(t)} I_{ii} \right) \]  

(8)

The matrices \( M(\nu_v(t)) \) are composed of the transition probabilities characterizing the push model, in terms of \( W(t) \) and \( Q(t) \), and of the pull transition matrix \( S(t) \). For this reason we will call the matrices \( M(\nu_v(t)) \) the mixed push-pull (MPP) matrices. Notice that for \( i \notin \nu_v(t) \) the \( i \)-th
row of \( M(\nu_r(t)) \) equals the \( i \)-th row of the push transition matrix \( P \). In the particular case \( \nu_r(t) = \{ \} \) holds that \( M(\nu_r(t)) = P(t) \).

In the situation of \( k \) homogenous personnel groups, the number of MPP-matrices \( M(\nu_r(t)) \) \( (\nu_r(t) \subseteq \{1,\ldots,k\}) \) equals \( \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \).

An MPP-matrix \( M = (I - W)\left[ \left( I + \sum_{i=\nu_r} I_{ii} \right) Q - \sum_{i=\nu_r} I_{ii} \right] = (m_{ij}) \) satisfies the following properties:

**Property 1.** The non-diagonal elements are between 0 and 2: \( 0 \leq m_{ij} \leq 2 \ \forall i \neq j \)

Property 1 let us conclude that an MPP-matrix satisfies the conditions to be an ML-matrix [32]. However an MPP-matrix has some additional characteristics:

**Property 2.** The diagonal elements are between -1 and 1: \( -1 \leq m_{ii} \leq 1 \ \forall i \)

**Property 3.** All the row sums are between 0 and 1: \( 0 \leq \sum_{j=1}^{k} m_{ij} \leq 1 \ \forall i \)

**Proofs.**

Since \( M = (m_{ij}) = (I - W)\left[ \left( I + \sum_{i=\nu_r} I_{ii} \right) Q - \sum_{i=\nu_r} I_{ii} \right] \) and denoting Kronecker delta by \( \delta_{ij} \),

for \( i \in \nu_r : \)
\[
m_{ij} = (1 - w_i)q_{ij} + (S,Q)_{ij} - \delta_{ij} \quad \text{and} \quad \sum_{j=1}^{k} m_{ij} = (1 - w_i)\sum_{j=1}^{k} (S,Q)_{ij}
\]

for \( i \in \{1,\ldots,k\} \backslash \nu_r : \)
\[
m_{ij} = (1 - w_i)q_{ij} \quad \text{and} \quad \sum_{j=1}^{k} m_{ij} = 1 - w_i
\]

These expressions prove the properties 1, 2 and 3, as \( Q \) is a row-stochastic matrix and the pull transition probabilities satisfy \( 0 \leq \sum_{j=1}^{k} s_{ij} \leq 1 \ \forall i \) and \( 0 \leq s_{ij} \leq 1 \ \forall i,j \).
In Bartholomew et al. [2] the limiting behavior of the push model is studied under the assumption that $\sum_{j=1}^{k} p_{ij} < 1 \ \forall i$, i.e. the wastage probability $w_i$ is different from zero. Because it is impossible that wastage never occurs, manpower systems are open systems. Hence, this assumption is reasonable. However, it is not for all personnel groups a necessary condition for examining the convergence of the mixed push-pull manpower system. For the further analysis of the MPP-matrix $M$ in this paper, for each personnel group $i$ in which vacancies arise, it is sufficient to assume that either the personnel strategy does not prescribe that all vacancies are entirely filled by pull promotions or that the wastage probability in that group is different from zero.

**Assumption 1.**

$$\exists I \subseteq \nu : \ w_i \neq 0 \ \forall i \in \{1,\ldots,k\} \setminus I$$

$$\sum_{j=1}^{k} s_{ij} < 1 \ \forall i \in I$$

**Property 4.** If Assumption 1 is satisfied, then for an MPP-matrix $M$ holds:

$$\forall i : \ 0 \leq \sum_{j=1}^{k} m_{ij} < 1$$

**Proof.** The row sums of the matrix $M$ satisfy:

$$\sum_{j=1}^{k} m_{ij} = (1 - w_i) \sum_{j=1}^{k} (S.Q)_{ij} = (1 - w_i) \sum_{j=1}^{k} s_{ij} q_{ij} = (1 - w_i) \sum_{j=1}^{k} s_{ij} \ \text{for} \ i \in \nu$$

and

$$\sum_{j=1}^{k} m_{ij} = 1 - w_i \ \text{for} \ i \in \{1,\ldots,k\} \setminus \nu$$

This let us conclude under assumption 1 that $\sum_{j=1}^{k} m_{ij} < 1$.

Which proves property 4.
Lemma 1. If Assumption 1 is satisfied, the inverse matrix \((I-M)^{-1}\) exists.

Proof. The existence of the inverse matrix \((I-M)^{-1}\) will be proved by showing the independency of the rows of the matrix \(I-M\). Without losing generality we will prove that the first row of \(I-M\) can not be written as a linear combination of the other rows of this matrix:

A solution \((C_2 \ C_3 \ ... \ C_k)\) of the following system of \(k\) linear equations \(E_1 \ E_2 \ ... \ E_k\)

\[
\begin{align*}
    m_{11} - 1 &= C_2 m_{21} + ... + C_k m_{k1} \\
    m_{12} &= C_2 (m_{22} - 1) + ... + C_k m_{k2} \\
    \vdots &= \vdots \\
    m_{1k} &= C_2 m_{2k} + ... + C_k (m_{kk} - 1)
\end{align*}
\]

satisfies for each subset \(I \subseteq \{2, 3, \ldots, k\}\):

\[
\sum_{j \in I} m_{ij} = \sum_{j \in I} C_i \left( \sum_{j \in I} m_{ij} - 1 \right) + \sum_{j \in I} \sum_{j \in I} m_{ij} \quad (9)
\]

as a result of adding up all the equations \(E_j\) with \(j \in I\). Hereby the notation \(\bar{I}\) refers to the set \(I = \{2, 3, \ldots, k\} \setminus I\). Since \(0 \leq \sum_{j \in I} m_{ij} < 1\) (Property 4) and the non-diagonal elements \(m_{ij} \geq 0\) (\(\forall i \neq j\)) are non-negative (Property 1), for all \(i \in I\), the coefficient of \(C_i\) is strictly negative:

\[
\sum_{j \in I} m_{ij} - 1 < 0.
\]

For all \(i \in \bar{I} = \{2, 3, \ldots, n\} \setminus I\) the coefficient of \(C_i\) is positive: \(\sum_{j \in \bar{I}} m_{ij} \geq 0\) . The fact that the left member of equation (9), \(\sum_{i \in \bar{I}} m_{ij}\), is positive, results in the conclusion that for a solution \((C_2 \ C_3 \ ... \ C_k)\) of the system it is not possible that \(C_i > 0\) for all \(i \in I\) and \(C_i \leq 0\) for all \(i \in \bar{I}\).

Since this reasoning holds for all subsets of indices \(I \subseteq \{2, 3, \ldots, k\}\), there can be concluded that the system has an empty solution set. For this reason the first row of \(I-M\) is not a linear combination of the other rows of this matrix. More generally, the rows of \(I-M\) are linear independent. For this reason \(I-M\) is a non-singular matrix for which the inverse matrix \((I-M)^{-1}\) exists.

Which proves the Lemma.
5. Asymptotic behavior of the mixed push-pull model

We investigate the asymptotic behavior of the mixed push-pull model under the assumption of time-homogeneity and a stabilized stock evolution.

**Assumption 2.** Time-homogeneity assumes time homogeneous transition probabilities, i.e. $Q(t)=Q$, $W(t)=W$ and $S(t)=S$ and a fixed recruitment policy $R(t)=R$ and $r(t)=r$. Furthermore, we assume that the desired personnel distribution $n^*(t)$ is fixed over time.

**Definition.** The evolution of the stock vector is called stabilized in case there exists a value $t^* \in \mathbb{N}$ such that $\forall t \geq t^* : \nu_+(t) = \nu_+(t^*)$.

Remark that in case $\forall t \geq t^*$ holds that $\nu_+(t) = \nu_+(t^*)$, the sequence $(M(t))$ converges for $t \to +\infty$ to the matrix $M = M(t^*)$.

**Assumption 3.** The evolution of the stock vector $n(t)$ is stabilized with $\lim_{t \to +\infty} M(t) = M$.

**Theorem 2.** If Assumptions 1-3 are satisfied and $\lim_{t \to +\infty} n(t) = n_e$ then

$$n_e = \left( n^* \cdot \sum_{i \in \nu_+} I_{ii} \cdot A + Rr \right) (I - M)^{-1} \quad (10)$$

**Proof.**

Since the evolution of the stock vector $n(t)$ is stabilized with $\lim_{t \to +\infty} M(t) = M$ there exists a value $t^* \in \mathbb{N}$ such that $\forall t \geq t^* : n(t) = n(t-1)M + n^* \cdot \sum_{i \in \nu_+} I_{ii} A + Rr$ (Theorem 1).

In case the stock vector $n(t)$ converges towards the limiting distribution $n_e$, this limiting distribution equals

$$n_e = n_e M + n^* \cdot \sum_{i \in \nu_+} I_{ii} A + Rr \Leftrightarrow n_e = \left( n^* \cdot \sum_{i \in \nu_+} I_{ii} \cdot A + Rr \right) (I - M)^{-1}$$

According to Lemma 1 the inverse matrix $(I - M)^{-1}$ exists.
Which proves Theorem 2.

**Theorem 3.** If Assumptions 1-3 are satisfied and all the eigenvalues \( \lambda \) of \( M \) satisfy \( |\lambda| < 1 \), then

\[
\lim_{t \to +\infty} n(t) = \left[ n^* \cdot \sum_{i \in U_t} I_{ii} \cdot A + Rr \right] (I - M)^{-1}
\]

**Proof.**

According to Theorem 1 there exists a value \( t^* \in \mathbb{N} \) such that \( \forall t \geq t^* : n(t) = n(t-1)M + n^* \cdot \sum_{i \in U_t} I_{ii} \cdot A + Rr \), since the evolution of the stock vector \( n(t) \) is stabilized with

\[
\lim_{t \to +\infty} M(\nu_1(t)) = M. \quad \text{For} \; M, \; \text{the inverse matrix} \; (I - M)^{-1} \; \text{exists according to Lemma 1.}
\]

For \( n_e = \left( n^* \cdot \sum_{i \in U_t} I_{ii} \cdot A + Rr \right) (I - M)^{-1} \) and \( \forall t > t^* \), the difference \( d(t) = n(t) - n_e \) satisfies:

\[
d(t) = n(t) - n_e = n(t-1)M + n^* \cdot \sum_{i \in U_t} I_{ii} \cdot A + Rr - n_e
\]

\[
= [d(t-1) + n_e]M + n^* \cdot \sum_{i \in U_t} I_{ii} \cdot A + Rr - n_e
\]

\[
= d(t-1)M + n_e M + n^* \cdot \sum_{i \in U_t} I_{ii} \cdot A + Rr - n_e
\]

\[
= d(t-1)M + n_e M + n^* \cdot \sum_{i \in U_t} I_{ii} \cdot A + Rr
\]

From the time \( t^* \) the evolution of the system is determined by the matrix \( M \) and therefore \( \forall t > t^* : d(t) = d(t^*)M^{t-t^*} \).

Under the assumption that all eigenvalues \( \lambda \) of \( M \) satisfy \( |\lambda| < 1 \) it holds that

\[
\lim_{t \to +\infty} M^{t-t^*} = 0
\]

and therefore: \( \lim_{t \to +\infty} d(t) = 0 \).

Consequently,

\[
\lim_{t \to +\infty} n(t) = n_e = \left( n^* \cdot \sum_{i \in U_t} I_{ii} \cdot A + Rr \right) (I - M)^{-1}
\]

Which proves Theorem 3.
**Theorem 4.** If Assumptions 1-3 are satisfied, then all the eigenvalues $\lambda$ of $M$ satisfy $|\lambda| \leq 1$.

**Proof.**

1. In case $\sigma_+ = \{\}$ no pull flows are involved and therefore the evolution is determined by the non-negative matrix $M = (I - W) \cdot Q = P$ with row sums less than 1. Consequently, the eigenvalues $\lambda$ of $P$ satisfy $|\lambda| \leq 1$ [32].

2. Pull flows are coming in the model in case for some $i$: $V_i(t) > 0$. Furthermore (according to (4)):

$$V_i(t) > 0 \iff n_i^* - n_i(t-1)(1 - w_i) > 0 \iff n_i(t-1) < \frac{n_i^*}{1 - w_i}$$

As proved in Theorem 3, the stabilized evolution of a stock vector is governed by the powers $t^i M$ of an MPP-matrix $M$. The properties of the powers $t^i M$ are determined by $|\lambda|^i$, for $\lambda$ eigenvalues of $M$. In case the involved MPP-matrix $M$ would have an eigenvalue with modulus greater than 1, the restriction $n_i(t-1) < \frac{n_i^*}{1 - w_i}$ can not be fulfilled for all values of $t$ and $i$. Consequently, under the condition that the evolution of the stock vector is stabilized, for all the eigenvalues $\lambda$ of $M$ holds that $|\lambda| \leq 1$.

Which proves Theorem 4.

Theorem 4 results in a necessarily condition to have an evolution of the stock vector $n(t)$ that is stabilized. Namely for an MMP-matrix $M(\nu_+ (t))$ with at least one eigenvalue $\lambda$ satisfying $|\lambda| > 1$ it is not possible to have value $t^* \in \mathbb{N}$ such that $\forall t \geq t^*$: $\nu_+(t) = \nu_+(t^*)$.
6. Numerical illustration

In this section, we provide a numerical illustration of the results in this paper. It concerns an organization in which the manpower system is divided in three homogeneous groups. The identification of the homogeneous groups is done based on the methodology in previous work [3-5]. Because of uncertain sales in the unpredictable organizational environment, the company decides to fix the workforce demand over time as \( n^* = (180 \ 140 \ 90) \). The initial stock at \( t = 0 \) is given by: \( n(0) = (160 \ 120 \ 60) \). The fixed recruitment policy is given by \( Rr = (66 \ 4 \ 0) \).

As shown in Bartholomew et al. [2], based on the estimator for transition probabilities given by Anderson and Goodman [33], using an historical dataset, the time-homogeneous wastage and push transition probabilities can be estimated. For the organization under study, the transition probabilities are given by:

\[
P = \begin{pmatrix} 0.71 & 0.1 & 0 \\ 0 & 0.7 & 0.2 \\ 0 & 0.35 & 0.52 \end{pmatrix}.
\]

We use equation (2) to reformulate this transition matrix \( P \) in terms of the parameters of the mixed push-pull model:

\[
Q = \begin{pmatrix} 0.88 & 0.12 & 0 \\ 0 & 0.77 & 0.23 \\ 0 & 0.4 & 0.6 \end{pmatrix} \quad \text{and} \quad I - W = \begin{pmatrix} 0.81 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.87 \end{pmatrix}.
\]

Notice that wastage is expected in every group of the system, such that Assumption 1 is satisfied. Further, the future personnel dynamics depends on firms’ promotion policy that regulates the pull transitions in the manpower system. To illustrate this, we study two different scenarios.
SCENARIO 1. Assume that, besides the push transitions, the organization sets its promotion policy by a pull approach. In case the stock in a personnel group is less than the desired number of employees, vacancies arise which are filled according to the promotion and recruitment policy characterized by:

$$S = \begin{pmatrix}
0 & 0 & 0 \\
0.85 & 0 & 0 \\
0 & 0.4 & 0
\end{pmatrix}.$$  

While vacancies in group 1 are entirely filled by external recruitments, vacancies in group 2 and 3 are partly filled by promotions from respectively group 1 and 2. Furthermore, notice that the mixed push-pull model is fully determined by time homogeneous transition probabilities. As a result, besides Assumption 1, also Assumption 2 holds. This allows us to illustrate the theoretical results of this paper concerning the asymptotic behavior of the mixed push-pull model.

For \( k = 3 \), the personnel dynamics in the mixed push-pull model can be regulated by eight possible transition matrices \( M(\nu_s(t)) \). Given \( Q, S \) and \( I - W \), by applying (8), an overview of the matrices \( M(\nu_s(t)) \) is given in Table 1. It shows that all eigenvalues of the matrices \( M(\nu_s(t)) \) are real and between -1 and 1. Consequently, according to Theorem 2, in case the evolution of the stock vector is stabilized, the limiting structures can be computed by (10). The resulting limiting vectors \( n_s(\nu_s(t)) \) are gathered in Table 1.

By applying (7), we compute the evolution of the stocks. Table 2 shows the extrapolation results, given the initial stock vector \( n(0) = (160 \ 120 \ 60) \). In the first and second time interval, by (6), the vacancies in each group are computed. Since \( V(1) > 0 \) and \( V(2) > 0 \), we know that \( \nu_s(1) = \nu_s(2) = \{1,2,3\} \). In time period [2,3], personnel mobility is no longer regulated by pull transitions to group 2, since \( V_2(3) = 0 \) and \( \nu_s(3) = \{1,3\} \). Subsequently, as can be seen in Table 2, the evolution of the stock vector is stabilized, since \( \forall t \geq 4 : \nu_s(t) = \nu_s(4) = \{3\} \). Hence, since Assumptions 1-3 hold, from Theorem 3, we know that the stock converges to \( n_s(\{3\}) = (232 \ 192 \ 96) \). As the results in Table 2 show, indeed, for \( t \geq 23 \): \( n(t) = (232 \ 192 \ 96) \).
Table 1. Stabilized stock evolution for every possible transition matrix $M(t_\nu(t))$

<table>
<thead>
<tr>
<th>$t\nu(t)$</th>
<th>$M(t_\nu(t))$</th>
<th>Eigenvalues</th>
<th>$n_\nu(t)$</th>
</tr>
</thead>
</table>
| $\{1,2,3\}$ | \[
\begin{pmatrix}
-0.10 & 0.10 & 0 \\
0.68 & -0.12 & 0.21 \\
0 & 0.62 & -0.27
\end{pmatrix}
\] | $\lambda_1 = 0.28$  
$\lambda_2 = -0.62$  
$\lambda_3 = -0.15$ | (229 161 91) |
| $\{1\}$ | \[
\begin{pmatrix}
-0.10 & 0.10 & 0 \\
0 & 0.70 & 0.21 \\
0 & 0.35 & 0.52
\end{pmatrix}
\] | $\lambda_1 = 0.9$  
$\lambda_2 = -0.1$  
$\lambda_3 = 0.33$ | (225 172 76) |
| $\{2\}$ | \[
\begin{pmatrix}
0.71 & 0.10 & 0 \\
0.68 & -0.12 & 0.21 \\
0 & 0.35 & 0.52
\end{pmatrix}
\] | $\lambda_1 = 0.81$  
$\lambda_2 = -0.27$  
$\lambda_3 = 0.58$ | (243 158 69) |
| $\{3\}$ | \[
\begin{pmatrix}
0.71 & 0.10 & 0 \\
0 & 0.70 & 0.21 \\
0 & 0.62 & -0.27
\end{pmatrix}
\] | $\lambda_1 = 0.82$  
$\lambda_2 = -0.39$  
$\lambda_3 = 0.71$ | (232 192 96) |
| $\{1,2\}$ | \[
\begin{pmatrix}
-0.10 & 0.10 & 0 \\
0.68 & -0.12 & 0.21 \\
0 & 0.35 & 0.52
\end{pmatrix}
\] | $\lambda_1 = 0.63$  
$\lambda_2 = -0.41$  
$\lambda_3 = 0.08$ | (227 157 69) |
| $\{1,3\}$ | \[
\begin{pmatrix}
-0.10 & 0.10 & 0 \\
0 & 0.70 & 0.21 \\
0 & 0.62 & -0.27
\end{pmatrix}
\] | $\lambda_1 = 0.82$  
$\lambda_2 = -0.39$  
$\lambda_3 = -0.1$ | (225 189 96) |
| $\{2,3\}$ | \[
\begin{pmatrix}
0.71 & 0.10 & 0 \\
0.68 & -0.12 & 0.21 \\
0 & 0.62 & -0.27
\end{pmatrix}
\] | $\lambda_1 = 0.8$  
$\lambda_2 = -0.58$  
$\lambda_3 = 0.11$ | (254 164 91) |
| $\{}$ | \[
\begin{pmatrix}
0.71 & 0.10 & 0 \\
0 & 0.70 & 0.21 \\
0 & 0.35 & 0.52
\end{pmatrix}
\] | $\lambda_1 = 0.9$  
$\lambda_2 = 0.33$  
$\lambda_3 = 0.71$ | (232 177 78) |
### Table 2. Extrapolation Scenario 1

| $t$ | $n(t)$ | $V(t)$ | $\nu_\sigma(t)$ |
|-----|--------|--------|-----------------
| 0   | 160    | 120    | 60              |
| 1   | 208    | 140    | 91              | 50   | 31   | 38   | $\{1,2,3\}$ |
| 2   | 217    | 161    | 87              | 12   | 13   | 11   | $\{1,2,3\}$ |
| 3   | 225    | 163    | 92              | 4    | 0    | 15   | $\{1,3\}$   |
| 4   | 227    | 169    | 91              | 0    | 0    | 10   | $\{3\}$     |
| 6   | 229    | 176    | 93              | 0    | 0    | 9    | $\{3\}$     |
| 8   | 230    | 181    | 94              | 0    | 0    | 9    | $\{3\}$     |
| 10  | 231    | 184    | 95              | 0    | 0    | 8    | $\{3\}$     |
| 12  | 231    | 187    | 95              | 0    | 0    | 7    | $\{3\}$     |
| 14  | 231    | 189    | 95              | 0    | 0    | 7    | $\{3\}$     |
| 16  | 231    | 190    | 96              | 0    | 0    | 7    | $\{3\}$     |
| 18  | 232    | 191    | 96              | 0    | 0    | 7    | $\{3\}$     |
| 20  | 232    | 191    | 96              | 0    | 0    | 7    | $\{3\}$     |
| 22  | 232    | 191    | 96              | 0    | 0    | 6    | $\{3\}$     |
| $\geq 23$ | 232 | 192 | 96 | 0 | 0 | 6 | $\{3\}$ |
SCENARIO 2. Assume that the organization decides not to use pull flows in its personnel strategy. Analogue to scenario 1, since for the eigenvalues $\lambda$ holds that $|\lambda|<1$, we know that if the evolution of the stock vector is stabilized, the limiting stock vector exists (Theorem 3). Since pull transitions are not possible in this example, by definition, the evolution of the stock vector is stabilized and $\nu_r(t)=\{\}$. By applying equation (10), the limiting stock is computed in Table 1 and is given by $(232\ 177\ 78)$. The extrapolation results, computed by (3), are given in Table 3. It shows indeed that for $t \geq 82$: $n(t) = (232\ 177\ 78)$.

Since no pull transitions intervene, scenario 2 is in fact an illustration of the traditional time-homogeneous push model with known recruitment, for which it is very well known that $\lim_{t \to +\infty} n(t) = R.r.(I - P)^{-1}$. Indeed, it is easily seen by (8) that $M(\nu_r(t)) = [I - W]Q$.

Remark that according to (1), in both scenarios and in every time interval, enough employees are available in the states for filling the vacancies by the proposed pull transitions, in order to meet the restriction of the mixed push-pull model.
Table 3. Extrapolation Scenario 2

<table>
<thead>
<tr>
<th>$t$</th>
<th>$n(t)$</th>
<th>$\bar{V}(t)$</th>
<th>$\nu_\tau(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>160</td>
<td>120</td>
<td>60</td>
</tr>
<tr>
<td>1</td>
<td>181</td>
<td>124</td>
<td>56</td>
</tr>
<tr>
<td>2</td>
<td>195</td>
<td>127</td>
<td>55</td>
</tr>
<tr>
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<td>206</td>
<td>131</td>
<td>55</td>
</tr>
<tr>
<td>4</td>
<td>213</td>
<td>134</td>
<td>56</td>
</tr>
<tr>
<td>6</td>
<td>222</td>
<td>141</td>
<td>58</td>
</tr>
<tr>
<td>8</td>
<td>227</td>
<td>147</td>
<td>61</td>
</tr>
<tr>
<td>10</td>
<td>229</td>
<td>152</td>
<td>64</td>
</tr>
<tr>
<td>15</td>
<td>231</td>
<td>162</td>
<td>69</td>
</tr>
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</tr>
<tr>
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<td>232</td>
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</tr>
<tr>
<td>60</td>
<td>232</td>
<td>177</td>
<td>77</td>
</tr>
<tr>
<td>80</td>
<td>232</td>
<td>177</td>
<td>77</td>
</tr>
<tr>
<td>81</td>
<td>232</td>
<td>177</td>
<td>77</td>
</tr>
<tr>
<td>≥82</td>
<td>232</td>
<td>177</td>
<td>78</td>
</tr>
</tbody>
</table>
7. Discussion and conclusions

In previous work on the time-homogeneous mixed push-pull manpower model, the limiting behavior was studied under the assumptions that push as well as pull flows are involved for (1) each state and (2) each time interval. The main contribution of this paper is that the asymptotic behavior of this model is studied by relaxing both assumptions from previous work. In present work, we studied manpower systems for which the intervening transition mechanism can be different according to the state. We examined the limiting behavior for manpower systems having a stabilized evolution of the stock vector. We showed that the evolution of the stock vector can only be stabilized with an MMP-matrix $M(\nu_c(t))$ for which each eigenvalue $\lambda$ satisfies $|\lambda| \leq 1$. Moreover it is proved that in case $|\lambda| < 1$ the stock vector converges. An analytical form for the limiting stock vector is found.

Although this paper makes a valuable contribution to the understanding of the asymptotic behavior of the mixed push-pull model, we need to consider some limitations that could be addressed in further research. First, we have shown that the manpower system converges if the evolution of the stock vector is stabilized and all eigenvalues $\lambda$ of the involved MPP-matrix have modulus $|\lambda| < 1$. We proved in this paper that the evolution of a stock vector cannot be stabilized with a specific matrix $M(\nu_c(t))$ in case for at least one eigenvalue $\lambda$ holds that $|\lambda| > 1$. However, it is not yet known what sufficient conditions the manpower system should satisfy in order to have a stabilized evolution. The convergence has to be studied in the situation a stock vector is stabilized and the involved MMP-matrix has some eigenvalues $\lambda$ with $|\lambda| = 1$.

Second, as seen in the second example, it is very well known that for the traditional time-homogeneous push model with known recruitment, the limiting behavior is independent from the initial stock. It is not yet clear that this is also the case in the mixed push-pull model. As the initial stock determines whether or not in the early time intervals the pull mechanism intervenes in one or more states, it has an impact on the future personnel mobility. Hence, the initial state strongly influences the early evolution of the stocks. However, this influence might reduce and finally disappear as time goes by.
References


