Sorting and ranking policies according to multiple indicators

P.L. Kunsch Vrije Universiteit Brussel BUTO Dept.
Pleinlaan 2, BE-1050 Brussel
pkunsch@vub.ac.be

Abstract. This article presents an easy-going and robust procedure for globally sorting policies given multiple independent indicators measuring achievement objectives set by policy-makers. Policies are processed on a stand-alone basis. In a first step a multiple rating is given to each policy by assigning it to merit classes, according to the way it fulfills the indicator objectives. In a second step, the indicators are in turn sorted out into priority classes indicated by the policy-makers. Finally, a global merit score and its associated probability distribution are obtained by combining these two results. Numerical experiments compare different combination formulas and conclusions are drawn about the proposed computing procedure. The ranking of several policies is obtained from the probability distributions of the global merit scores.

Keywords: policy, multiple indicators, merit classes, priority classes, merit row vector, priority row vector, global merit score, global-score probability distribution, trade-off factor, ranking matrix

1. Introduction

In many decision-making processes with multiple criteria (MC) hard data may be completely missing, or they are provided by decision-Makers (DMs) as semantic judgements, such as ‘Excellent’, ‘Good’, ‘Unacceptable’, etc., or by means of confidence intervals, fuzzy sets, ordinal rankings, etc. Priorities on the criteria are addressed by qualifiers, such as Very important’, ‘Rather marginal’, etc. The impreciseness stems from the data sources. For example, in the choice of an energy mix in a country, policies foresee different proportions of coal, natural gas, nuclear power, wind energy, etc. The merits of the policies is based on the evaluations of several indicators, such as costs, safety of supply, pollution, and others obtained by simulation or optimisation, or from more or less qualitative judgement on the fitness of particular decisions with respect to objectives. In addition the set of alternatives is evolving in the continuous search for still better decisions. In such situations it is necessary to evaluate each decision separately: a global assessment, combining all multiple attributes is provided on a stand-alone basis. Sorting out in merit classes is the way proposed in this article for global

MC assessments. Rankings may be considered at a later stage when a more complete alternative portfolio is available.

For evaluating, sorting and ranking alternatives two main MC families are common (Vincke, 1992); (Figuera et al., 2005): either based on pair-wise comparisons, or on utility theory. MC methodologies based on pair-wise comparisons are, just to name the most popular ones: ELECTRE (Roy, 1996); PROMETHEE (Brans and Mareschal, 2005); Analytical Hierarchical Process AHP (Saaty, 1980); MACBETH (Bana e Costa and Vansnick, 1994). Many Multi-attribute Utility Theory (MAUT) methodologies exist in the literature that cannot be reviewed here. MAUT is more adapted to the type of problems just described, as the global utility of each alternative is evaluated on a stand-alone basis. The SMART method (Edwards and Barron, 1994) is the simplest one, consisting in a weighed sum of utilities, each picked up in a [0, 100] interval. How to choose the weights is not made explicit, and choices of utility values are left to subjective judgements. The more sophisticated Stochastic Multiobjective Acceptability Analysis (SMMA) (Lahdelma, et al. 1998) explores the weight space by means of Monte Carlo simulations, and provides statistical results on the weights. Other MAUT techniques require quite some more assumptions on the input data and independence properties of attributes; in the stochastic case risk attitudes come into play to elaborate utility functions (Keeney and Raiffa, 1976). In (Kunsch, 2009, 2010) the ranks of n alternatives by criterion are identified with not-normalised utilities. Each ranking by criterion is placed in a *criterion rank vector* in the $\mathbb{R}^n$ vector space. The global *n-D* rank vector is a linear combination with non-negative coefficients of the resulting K criterion rank vectors, i.e., a weighed vector sum in $\mathbb{R}^n$. DMs are requested to rank the criteria by priority, producing herewith order statistics (Arnold et al., 1992) used for Monte-Carlo computations on the global ranks. The same approach is shown to be applicable when the evaluations are scores on a m-point scale, common to all criteria. (Kunsch, 2012) illustrates that the score approach is free of common failures observed in pair-wise comparisons methods, particularly PROMETHEE (Brans and Mareschal, 2005).

The purpose of the present article is to present a full stand-alone procedure in the case described above, going beyond purely ordinal-rank data, and avoiding anomalies observed in pair-wise comparison methods. In a first stage of the decision process sorting out is the keyword. Merit evaluations are made on a stand-alone basis for each alternative, called *policy* in the following. Furthermore DMs, called later on *policy makers* (PMs), are asked to assign by criterion a given policy to a *merit class* according to the way it fulfils the objectives of the criterion, called *indicator* in the following. In the next step each indicator is assigned to a *priority class* according to its importance in the policy-making process. Both sorting-out processes of policy and indicators are then combined for sorting out the policy into a *global merit class*. Ranking may take place once a policy portfolio has been constituted. The article is organised as follows:

In section 2, the methodological framework is set out by defining the terminology, merit classes by indicator, and priority classes of indicators. Global merit classes are defined in section 3, and the proposed computing procedure of global merit scores is established. In section 4 several numerical experiments are performed in order to illustrate the feasibility and robustness of the proposed procedure, and to compare different computing schemes. In section 5 the ranking procedure within a policy portfolio is introduced. Conclusions are given in section 6 along with indications about possible future research work.

2. Methodological framework

2.1 Terminology and context

The terminology is first discussed in support of the context:

(1) ‘Policy’ better suggests than ‘Alternative’ the structural and time-based dynamic nature of decisions to be made under the conditions of uncertain future;
(2) ‘Indicator merit’ better suggests than ‘criterion’ the composite nature of policies;
(3) ‘Indicator priority’ is used instead of weights, trade-offs, or preferences to better stress the often verbal nature of statements on the relative importance of indicators;
(4) ‘Global merit’ betters suggest than ‘aggregated preferences’ the process of finding a global merit score without the help of pair-wise comparisons of policies.

Expressed differently: firstly, policy-making is not a well-cut process with well-defined static data but a complex dynamic process with many entangled aspects, and imprecise data. For designing public policies or corporation strategies, for example, data stem from projections towards the future and are uncertainty fraught. Secondly, since the policy portfolio is evolutionary, there is no other way than rating each policy for its own merits against the indicators. Sorting out into merit classes is herewith given priority over ranking. Three steps followed in sequence are listed, and the corresponding sub-section is indicated in Table 1.

Table 1. Three steps for policy sorting out into global merit classes

Sorting-out steps	**Description**	**Sub-section**
I Merit classes of policy by indicator	a) Defining merit classes b) Assigning the policy to a merit class	2.2
II Priority classes of indicators	a) Defining priority classes b) Assigning each indicator to a priority class	2.3
III Global merit classes of policy	a) Principles of the assignment to a global merit class b) Global assignment procedure	3.1 3.2

2.2. Definition of merit classes and merit row vector

Assume that the set of indicators $C_k, k \in \{1, 2, ..., K\}$ to take the gauge of policies is available and complete. The preparation step of pertinent and independent indicators is an important one, but it is beyond the scope of this article.

In the first task achievement objectives for each indicator are defined; several performance echelons are introduced for assessing policy achievements. The human mind is limited when comparing too many echelons. The 7 ±2 golden rule makes sense (Miller, 1956). In the following it is assumed throughout that five echelons will be sufficient in many cases. This gives the following set of five semantic and numerical labels for the five acceptable merit classes:

$$\{Best = 1, Rather\ Good = 2, Average = 3, Rather\ Poor = 4, Just\ Acceptable = 5\} \cup \{Unacceptable\} \quad (1)$$

Take as an example in Table 2 the percentage of the Belgian electricity demand in 2030, not covered by domestic production, but by importations from neighbouring countries. The government defines echelons for this percentage, so that policies are classified according to their safety-of-supply performances.

This 5+1-echelon setting is common; the *German Grading Scale* is an example: it has been used for centuries as rating instrument in German-speaking schools and universities. In the standard version integer marks for different disciplines are given according to the fulfillment of the corresponding study objectives. The best mark is ‘1’ corresponding to the ‘A’ grade in USA. The last echelon ‘6’ is eliminatory, so that the corresponding policy is discarded for the further assessments, thus compensation by acceptable grades is not allowed. Herewith an inverse Likert’s 5-point scale is obtained: the numerical class label is the merit score of the policy with respect to the indicator objectives. It is sensible to use the best possible score to be ‘1’ as being the best position as in ordinal ranking.

A second task is to map Policy *P* onto one merit class only of the *k-th* indicator. Given $S_k^P \in \{1, 2, 3, 4, 5\}$ the set of numerical merit-class labels, and the actual *P* performance, the merit mapping[1] is:

$$\mathrm{M}_k : P \to S_k^P \quad (2)$$

This defines the *merit row vector* S^P of policy P for *K* indicators:

$$S^P = \left(S_1^P, S_2^P, ..., S_K^P\right) \quad (3)$$

<u>*Example:*</u>

Consider five energy mix policies and a classification by the electricity safety-of-supply indicator according to Table 2; the percentage column vector of

[1] MACBETH (Bana e Costa and Vansnick, 1994) generates iteratively a consistent numerical scale, i.e., respecting transitivity, by means of qualitative pair-wise judgements on policies. This approach is thus useful for assisting the definition of merit-class echelons by indicator and the mapping (2), using reference policies as benchmarks.

importation rates of the five policies in arbitrary order is estimated as (11%, 5.5%, 23%, 8.2%, 3.1%)', the merit column vector the five policies is then (4, 2, 5, 3, 2)'; no policy gets discarded, but none belongs to the class 'Best=1'.

Table 2. 5+1 echelons rating the domestic electricity safety of supply in 2030, expressed as a percentage of imported electricity S.

Class n°=score	Performance S (%)
1	$S < 3$
2	$3 \le S < 7$
3	$7 \le S < 10$
4	$10 \le S < 20$
5	$20 \le S < 30$
Unacceptable	$S \ge 30$

2.3 Definition of priority classes and priority row vector

Consider now the set of *K* indicators, $\{C_k, k = 1, 2, ..., K\}$, their objectives, and the merit row vector of the policy after the mapping (2). People are presumably not able to give crisp weights in one go: assigning an importance label to each indicator is certainly easier. The *7 ±2* golden rule again makes sense: *5* classes are chosen in parallel with (1):

$$\{Most\ I = 5, Rather\ I = 4, Moderately\ I = 3, Rather\ Less\ I = 2, Marginally\ I = 1\} \cup \{Negligible = 0\} \quad (4)$$

where 'I' means 'Importance'. The following priority mapping is performed:

$$\Psi_k : C_k \rightarrow X_k \ \exists k \neg X_k = 5 \quad (5)$$

where $X_k \in \{5, 4, 3, 2, 1\}$ labels the importance class to be chosen for the *k-th* indicator, remembering that negligible indicators are ignored. The condition on the right of (4) indicates that at least one indicator has the highest priority, being most important. Let us define the *priority row vector*:

$$X = (X_1, X_2, ..., X_K)\ \exists k \neg X_k = 5 \quad (6)$$

Using both merit row vectors and priority row vector gives us the means for sorting out some policy into global merit classes, and obtaining herewith a scalar *global merit score*.

3. Global merit classes

3.1 Principles

Consider policy *P, K* indicators, and the merit and priority row vectors *X*, S^P respectively, as defined in section 2:

$$P : (X, S_P)\ with\ X = \{X_k, k = 1, 2, ..., K\};\ S^P = \{S_k^P, k = 1, 2, ..., K\} \quad (7)$$

The global-sorting task consists in the global rating of *P*, by assigning it to one, or more than one, global merit class(es) (1). Call G^P the set of global-class numerical labels, and define the mapping φ:

$$\varphi : P \rightarrow G^P\left(X, S^P\right) \subset \{1,2,3,4,5\};\, \mu^P \qquad (8)$$

where $\mu^P = \left\{\mu_i^P \; i = 1,2,...,5\right\}$ is the set of *possibility grades* (PG) representing the *possibility distribution* (Buckley et al., 2002) of P on the merit classes: μ_i^P is the PG of *P* belonging to the *i-th* merit class (1):

$$\begin{aligned} &\mu_i^P = 0 && if\ i \notin G^P \\ &\mu_i^P = 1 && if\ G^P = \{i\} \\ &0 < \mu_i^P < 1 && if\ 1 < card\left(G^P\right) \leq 5 \end{aligned} \qquad (9)$$

The possibility distribution μ^P is normalised, i.e., it sums up to one for all five merit classes to which P may belong: μ^P has the same properties as a discrete probability distribution:

$$\sum_{i=1}^{5} \mu_i^P = 1 \qquad (10)$$

For the sake of easy language μ^P will be handled in the following as *probability distribution*, rather than as possibility distribution. A probabilistic interpretation is that during many repeated processes of globalising indicator scores, the policy P is assigned to the i-th global merit class with the μ_i^P % frequency.

The φ mapping (8) is now further elaborated to make it operational. The same merit scale is used for all indicators, so that the scores in the merit row vector are commensurable and combinable by arithmetical operations, additions and multiplications, to obtain a natural number in the $\{1,2,3,4,5\}$ set. The task is to compute one or several integer values from this set to populate G^P in (8) by linearly combining the indicator scores in the merit row vector. Calling $g_l^P \in G^P \; l = 1,2,...,card\left(G^P\right)$ one integer global score, one thus computes:

$$1 \leq g_l^P = \sum_{k=1}^{K} x_k S_k^P \leq 5 \quad g_l^P \in \mathbb{N} \qquad (11)$$

where $\mathbb{N}$ is the set of natural number, and x_k^P's are strictly positive coefficients, assuming no zero-priority indicator. The following property result for g_l^P:

$$\min\left(S_k^P\right) \leq I_{\min}^P = \left\lfloor \sum_{k=1}^{5} x_k^P S_k^P \Big/ \sum_{k=1}^{5} x_k^P \right\rfloor \leq g_l^P \leq I_{\max}^P = \left\lceil \sum_{k=1}^{5} x_k^P S_k^P \Big/ \sum_{k=1}^{5} x_k^P \right\rceil \leq \max\left(S_k^P\right) \qquad (12)$$

i.e., the global integer score g_l^P has one or two integer values: they are obtained by normalising to one the sum of the coefficients in the linear combination, taking the closest integer values downwards $\lfloor\ \rfloor$ or upwards $\lceil\ \rceil$ in the $[1,5]$ range. Clearly the linear coefficients are rational strictly positive numbers, i.e., they are representable as follows:

$$x_k^P = W_k / Q \in \mathbb{Q}^+; \quad W_k, Q \in \mathbb{N} \qquad (13)$$

The coefficients x_k^P are then:

$$x_k^P = g_l^P W_k \Big/ \sum_{m=1}^{K} S_m^P W_m \rightarrow Q = \sum_{m=1}^{K} S_m^P W_m \Big/ g_l^P \qquad (14)$$

<u>Proof</u>

$$\sum_{k=1}^{K} x_k^P S_k^P = g_l^P \left[\sum_{k=1}^{K} W_k S_k^P \Big/ \sum_{m=1}^{K} S_m^P W_m \right] = g_l^P \qquad (15)$$

<u>Example</u>

$$K = 5 \quad S^P = \{1,2,2,4,3\} \quad W = \{5,5,4,3,2\}$$

$$\sum_{k=1}^{5} x_k S_k^P \Big/ \sum_{k=1}^{5} x_k = \sum_{k=1}^{5} W_k S_k^P \Big/ \sum_{k=1}^{5} W_k$$

$$= (5\times1+5\times2+4\times2+3\times4+2\times3)/(5+5+4+3+2) = 41/19 = 2.16 \qquad (16)$$

$$\rightarrow G^P = \{g_1^P = 2, g_2^P = 3\}; \mu^P = \{\mu_2^P = 0.84, \mu_3^P = 0.16, \mu_1 = \mu_4 = \mu_5 = 0\}$$

$$x_1^P (g_1^P = 2, \mu_1^P = 84\%) = (W)/(Q = 41/2) = (10/41, 10/41, 8/41, 6/41, 4/41)$$
$$x_2^P (g_2^P = 3, \mu_2^P = 16\%) = (W)/(Q = 41/3) = (15/41, 15/41, 12/41, 9/41, 6/41) \qquad (17)$$

The priority row vector X containing numerical class labels is mapped onto $(W) = (W_k) \quad \forall k = 1,2,...,K : W_k \in \mathbb{N}$, which is the *trade-off row vector* to be used in the weighed sum; inverse trade-offs are $(W)/(W_{\max} = 5) = (1, 1, 4/5, 3/5, 2/5)$, the maximum trade-off is here 5/2=2.5. Because the indicator order in the row vector is not important, (W)'s components may be ordered in decreasing order, $W_{\max} = 5$ coming first.

3.2 Global assignment procedure

The proposed procedure for mapping policy P into a global merit class is now available; it proceeds in four steps. Remember the *m=5* choice of the inverse 5-points Likert scale for sorting out P achievements into merit classes, and of the direct 5-points Likert scale for sorting out indicators into priority classes[2]. In the following the Policy (P) superscript is left out in formulas.

Step 1 Sorting out into merit row vector
The five merit classes are labelled according to (1) as:

$$\{Best = 1, Rather\ Good = 2, Average = 3, Rather\ Poor = 4, Just\ Acceptable = 5\}$$

Any policy with class label=’Unacceptable’ has been discarded beforehand. The merit scores by indicators are chosen from the numerical label set.
The output of this step is defined as the *1xK merit row vector S*.

Step II Sorting out into priority row vector
The five classes are labeled as in (4) as follows:

$$\{Most\ I = 5, Rather\ I = 4, Moderately\ I = 3, Rather\ Less\ I = 2, Ma\,\mathrm{rg}\,inally\ I = 1\}$$

(I=Important)
Negligible indicators have been discarded. The output of this step is defined as the *1xK priority row vector X*.

Step III Mapping the priority row vector into the trade-off row vectors
This step consists in mapping the *1xK* priority row vector *X* into a trade-off row vector (W) populated with strictly positive numbers, defined up to an affine transformation. Picking up *K* natural numbers from sets like $W^{(5)} = \{5,4,3,2,1\}$; $W^{(100)} = \{1,\ 2,\ \ldots,\ 100\}$ are possible ways, in which the superscript indicates the maximum trade-off. At least one indicator must be given the largest number in the set, i.e.: 5, 100, etc.

The output of this step is defined as the *1xK trade-off vector* (W). In section 4 numerical experiments on several $X \rightarrow (W)$ mapping schemes are performed and compared.

Step IV Sorting out globally into five merit classes
A global rational number is obtained by calculating the scalar product of the merit row vector *S* and some (W) instance, and dividing it by the sum of (W) components; the two closest integers give two possible global merit scores for this (W) instance:

$$g_1 = \left\lfloor g = (W).S \Big/ \sum_{k=1}^{K} W_k \right\rfloor \quad g_2 = \left\lceil g = (W).S \Big/ \sum_{k=1}^{K} W_k \right\rceil \qquad (18)$$

Two probability values summing up to one of those global scores are:

[2] Should finer graining be necessary, either for merit scores, or for priorities, use $m \le 9$ according to the golden rule, and an odd number of points to have a class representing a medium value on the scale, thus $m = 7\ or\ 9$.

$$\mu_1 = g_2 - g \quad \mu_2 = g - g_1 \tag{19}$$

Example:

$S = (3,\ 2,\ 2,\ 1,\ 4)$; $(W) = (5,5,3,3,3)$:

$$\varphi : P \to g = 46/19 = 2.42 \tag{20}$$

$$G = \{g_1 = 2; g_2 = 3\}$$
$$\mu = \{0, 58\%, 42\%, 0, 0\} \tag{21}$$

By instance only two neighbouring global classes have non-vanishing probabilities. The probability distribution obtained for all considered (W) instances in the global-score evaluations gives the *global score-probability vector*: the five components in this distribution represent the probabilities for policy P to get the corresponding global merit score. How to choose $(W)'s$ instances is further discussed in the next section 4, by means of numerical experiments.

4. Numerical experiments

4.1 Planned numerical experiments for choosing (W) vectors

The choice of (W) vectors is the only unanswered question regarding the procedure in section 3. It is now discussed as several schemes are compared by numerical experiments. The rationale is to apply the schemes to many different combinations of merit and priority row vectors to allow analysing and explaining the observed differences, and to make recommendations about the usefulness of these schemes. Three representative experiments detailed below are performed for *K*=3, 6, 5 respectively. The enumeration of the different combinations in each exercise is done by application of group theory to the S_K symmetric group of permutations of the *K* first integers (Rotman, 1994).

a) In a first experiment for *K=3*, the global merit scores of all 1,385 different combinations of merit and priority row vectors are exhaustively calculated;
b) In a second experiment for *K=6*, priority row vectors of the type (5, 5, x, x, y, y) where $1 \le x \ne y < 5$ are considered in 20,250 different combinations with merit row vectors;
c) Finally in the third experiment for *K=5*, the merit row vector $[5\ 4\ 3\ 2\ 2]$ is considered, and all 2,101 Pareto-optimal combinations with priority row vectors are evaluated.

There are infinitely many ways of choosing (W)-vectors, because (W) ‘s components are defined up to affine transformations with positive multiplicative coefficients. Taking this into account, two different families of approaches either deterministic or stochastic may be considered, as follows:

A. In the *deterministic approach* K well defined values are given to the $(W)'s$ components, e.g., integer numbers in the range [1, 100] representing priorities (remember that no indicator has zero priority). $|G^P|=2$: G^P contains two adjacent integer global scores, and the global-score probability vector has two non-zero components;

B. In the *stochastic approach* rank-distributed random values are drawn to represent the priority row vector for a sufficiently large number of (W) instances, giving a global-score probability vector with up to five non-zero components.

Both deterministic and stochastic schemes are now closely examined.

A. Using determinitic $(W)'s$

An easy way is to use either directly the numerical label of the priority class, i.e., an integer in the 1-5 range, or an affine transformation of this integer giving strictly positive (W)'s components. At first sight, this seems to be a bit arbitrary: it is why some argumentation is needed for justification. Let us proceed as follows:

Consider an interval of values, e.g., $[1,100]$ from which 0 is excluded, because all indicators have some importance; call R the *trade-off ratio* between the least important indicator(s) and the most important indicator(s); accordingly $1/R$ is the maximum trade-off:

$$R = W_{\min}/W_{\max} > 0 \qquad (22)$$

Choose $W_{i_1} = 100 \quad i_1 = 1,..., p_i$, for the indicator(s) with the highest priority $W_{\max}$ with multiplicity p_i; choose $K - p_i$ values from the same interval for the remaining indicators with lower priorities. Choosing the class labels from the priority class numerical labels $\{5,4,3,2,1\}$ from (4) is clearly equivalent to choosing (W) '*s* components from the discrete set $\{100,80,60,40,20\} \triangleq W^{(5)} = \{5,4,3,2,1\}$ [3] with *R=1/5*. Why is this choice pertinent? No experimental testing has been conducted about how a majority of people would choose (W)'s components, but in the author's opinion the following intuitive declarations might be easily accepted by most people:

[3] The symbol $\triangleq$ indicates here the equivalence between two sets of numbers

1. In a [1, 100] interval, people will choose for *W's* 'round' integer numbers like 20, 35, etc., rather than 23, 34,18537, 68 1/7, $\sqrt{20}$, etc.;
2. People tend to distribute more or less equally $W's$ values representing the echelons measuring the relative importance of indicators. Therefore one would expect that the set $W^{(5)}$ with equidistant components is more plausible than other ones, with irregularly distributed components, like $W' = \{100, 95, 84, 63, 52\}$, or $W'' = \{100, 89, 78, 22, 3\}$ representing the five echelons (4).

The equivalent $W^{(5)}$ sets of five integers with maximum trade-off *1/R*=5 are not the only possible sets with equidistant components, given different $W_{\max}$ and *R*, other such sets are obtained by affine transformation, e.g.,

$$W^{(9)} = \{90, 70, 50, 30, 10\} = W^{(5)} - 10 \triangleq \{9, 7, 5, 3, 1\} \tag{23}$$

with[4] $R = 1/9$; a limiting case for $R \to 0$ in which marginal indicators are discarded, would be:

$$W^{(\infty)} = \{100, 75, 50, 25, 0\} = 10\,(W^{(5)} - 20)/8 \triangleq \{4, 3, 2, 1, 0\} \tag{24}$$

In the numerical experiments the three sets $W^{(5)}, W^{(9)}, W^{(\infty)}$ will be compared as representatives of the deterministic case. For keeping formulations short, choosing (W) components from any of those $W^{(*)}$ sets will simply be indicated as computing with the $W^{(*)}$ set.

B. Using random *(W)'s*

In contrast with the A case, the priority row vector contains declarative labels (4) at first, but random numbers are then assigned for computing. Rank statistics are suitable (Arnold et al., 1992): five random numbers from the [0, 1] interval are produced and ordered by decreasing values to represent priorities from 'Most Important' to 'Marginally Important'. The procedure for creating rank statistics is implemented as follows:

1) Set the counter to $i_R = 0$, and define its maximum value as $N_R >> 1$;
2) Generate a set of five uniformly distributed random numbers in the [0,1] interval; Order the five numbers from the largest to the smallest;

[4] The AHP methodology (Saaty, 1980) use the same $W^{(9)}$ set for criteria trade-offs; it could possibly be used for preparing consistent $(W)'s$ with integer components drawn from this set.

3) Divide the ordered numbers by the largest, and assemble the ratios into the row vector R_W; the first component being then equal to one; the last component is equal to *1/R*, the trade-off factor (22);
4) Increment the counter to $i_R + 1 \le N_R$; if $i_R < N_R$ move back to 2) otherwise stop.

N_R ordered row vectors R_W are herewith generated:

$$R_W^i = (r_{5:5}^i = 1, r_{4:5}^i, r_{3:5}^i, r_{2:5}^i, r_{1:5}^i) \quad i = 1, 2, ..., N_R \tag{25}$$

The component $r_{j:5}^i$ for the *i-th* instance represents the numerical priority given to the *j-th* class; the first class 'Most Important' gets 1; $(W)_i$ is defined accordingly.

<u>Example:</u>

$$X = (5,4,4,2,2) \rightarrow (W)_i = R_W^i = (1, r_{4:5}^i, r_{4:5}^i, r_{2:5}^i, r_{2:5}^i) \tag{26}$$

Order statistics from uniform distribution have statistical properties now described. For the assumed uniform distribution in [0, 1], $r_{j:5}^i$'s $1 \le j \le 4$ are instances of random variables $V_{j:5}$. Consider the Beta(a,b) distribution and its probability density:

$$f(x,a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \tag{27}$$

where Γ is the Gamma function with $\Gamma(n) = (n-1)!$, for positive integers *n*; the Beta-distribution mean is $a/(a+b)$ and its variance $ab/\left[(a+b)^2(a+b+1)\right]$. The following properties are demonstrated in (Arnold et al., 1992, pp. 78-79):

a) The random variables $V_{j:5}$ $j = 4,3,2,1$ are statistically independent;

b) $V_{j:5}$ is distributed according to *Beta(j,5-j),*with mean $\mu_{j:5}$ and variance $\sigma_{j:5}^2$ given in (28);

c) Beta(j,5-j) values are in $]0,1[$, and vanish in 0, and 1, meaning that infinite trade-offs have zero probability.

$$\mu_{j:5} = j/5 \quad \sigma_{j:5}^2 = j(5-j)/150 \quad j = 4,3,2,1 \tag{28}$$

As seen from (28), the mean values of the ordered vector are $\overline{R_W^i} \triangleq W^{(5)} = (5,4,3,2,1)$. This reinforces the idea brought forward above to favour the straight use of the numerical priority-class labels 1 to 5 in the deterministic A) case. In the following let us call $W^{(R)}$ the sets of ordered random numbers generated to populate random instances of $(W)'s$ as indicated.

4.2 Numerical comparisons

In this sub-section the three numerical experiments a), b), c) described in 4.1 are performed with MATLAB® by using the three sets of integer numbers $W^{(5)}, W^{(9)}, W^{(\infty)}$ as representative for the deterministic A) case, and the set $W^{(R)}$ with $N_R = 1.000$ instances of K rank-ordered values in $]0,1[$ for the stochastic B) case. Note that A) gives fast results with short computing times for all three deterministic set, while B) requires more time resources. Nevertheless in no experiment did computing take more than 10 min. on a 32-bit netbook under Windows 7. The following results have been obtained though not all are presented in detail:

- The global scores obtained for the described W sets;
- The difference between global scores obtained by $(W) \in W^{(5)}$, called later on *reference case* and other (W) *'s* generated by different $\left(W^{(*)}\right)$ sets;
- The global-score probability vectors corresponding to all combinations of priority and merit row vectors.

The following notations are used:

- z_5, z_9, z_∞ represent the global scores obtained from respectively $W^{(5)}, W^{(9)}, W^{(\infty)}$ sets;
- z_R is the mean global score obtained with $W^{(R)}$ sets for N_R*=1000* instances;
- $\Delta_{5*} = |z_5 - z_*|$;

<u>Experiment 1:</u>
K=3; all different combinations of priority and merit row vectors are considered. Both types of vectors are unchanged under the actions of the S_3 symmetric group of *3!=6* permutations of their columns (Rotman, 1994). The enumeration gives 1,385 so-called orbits, meaning different combinations; in detail, there are four classes of priority row vectors ordered from left to right by decreasing class labels. These classes correspond to the four ordered partitions of *K=3, i.e., 2+1, 1+2, 1+1+1, 3*:

- **1+1+1** gives one ‘5’ class and two classes with two different labels from{1,2,3,4}, e.g., (5, 2, 1); there are 6 priority row vectors in this class, and 5^3=125 merit row vectors to be combined, giving in total 6x125=750 different combinations;

- **2+1** gives two ‘5’ classes and one class with one label from {1,2,3,4}, e.g., (5,5,3), there are 4 priority row vectors in this class, and 75 merit row vectors to be combined, giving in total 4x75=300 different combinations;
- **1+2** gives one ‘5’ class and two classes with the same label from {1,2,3,4}, e.g., (5, 2, 2); there are 4 priority row vectors in this class, and 75 merit row vectors to be combined, giving in total 4x75=300 different combinations;
- **3** gives three ‘5’ classes, there is only one priority row vector (5,5,5), and there are 35 possible merit row vectors, and also 35 different combinations: the results are plain, as all indicators have equal priority; whatever the approach the global score is always equal to the average indicator score.

Computations have been performed separately for each partition. For reasons of symmetry, for each partition, the average global score over all different combinations is $\overline{z_*} = 3$, the average of 1 to 5 integer numbers. For the same reason, z -probability distributions for all $W^{(*)}$ sets are symmetrical around 3 for each partition. It is also observed that probability distributions are the same for both $W^{(5)}, W^{(R)}$ sets when computed over all combinations and many $W^{(R)}$ instances. This is due to the symmetry of the sets of different combinations and to the properties of the Beta distribution, which has zero-probability for infinite trade-offs. Remember however one difference for the stochastic $W^{(R)}$ set: the global-score probability distribution by instance and by combination may have up to five non-zero components, by contrast deterministic $W^{(*)}$ sets have two only, see (18). Due to the lack of space no detailed analysis of the global-score probability distributions per combination is provided here.

Table 3 gives the statistics of the differences $\Delta_{59}, \Delta_{5\infty}$ with the reference case for all *K=3* partitions. Figure 1 plots the global scores z_5, z_9, z_∞ for the (3=1+2) partition (the reference global scores z_5 are plotted in increasing order to improve visibility). In this partition the largest differences are observed, obviously for $\left(W^{(\infty)}\right)$-type vectors, because infinite trade-offs appear for vanishing priority. For example, $X = (5,1,1)$ produces the largest absolute differences when it is mapped into $\left(W^{(\infty)}\right) = (4,0,0)$, only the first component of the merit row vector being actually used. The mean global scores z_R obtained again for the (1+2) partition are shown in Figure 2. It is seen that $\left(z_5 - z_R\right)$

differences are hardly visible, being quite small indeed: a zoom on $(z_5 - z_R)$ is plotted in Fig. 3. Note the central symmetry in the graph: it explains why z_5 and z_R have the same global-score probability distribution over all combinations and instances. Most absolute z differences lie in the range 0 to 0.05. It can be verified that outliers in the range 0.10 to 0.15 correspond to priority row vectors of the type $X = (5,1,1)$ showing maximum trade-offs, so that integer values on the edge of the global merit scale appear more frequently in the probability distributions.

Table 3. *K*=3. Statistics of the absolute differences $\Delta_{5*} = |z_5 - z_*|$ for the deterministic W sets for all partitions. $\mu's$ are the means, $\sigma's$ the standard deviations, *L's* the maximum values.

Partititions of *K=3*	**#combinations**	**Representative priorities**	$\mu_{\Delta 59}$	$\sigma_{\Delta 59}$	$L_{\Delta 59}$	$\mu_{\Delta 5\infty}$	$\sigma_{\Delta 5\infty}$	$L_{\Delta 5\infty}$
1+1+1	750	(5,2,1)	0.05	0.05	0.27	0.14	0.13	0.70
1+2	300	(5,2,2)	0.06	0.08	0.42	0.17	0.22	1.15
2+1	300	(5,5,3)	0.03	0.03	0.15	0.07	0.07	0.36
3	35	(5,5,5)	0	0	0	0	0	0

Figure 1. K=3. Comparison of the global scores for the 1+2 partition $z_5(\times)$, $z_9(*)$, $z_\infty(+)$

Figure 2. K=3. Comparison of the global scores for the 1+2 partition $z_5(\times)$, $z_R(o)$ for $N_R = 1,000$

Figure 3. Zoomed differences from Fig. 2 of $(z_5 - z_R)$

The statistical analysis of the absolute differences between one unique z_5 score and $N_R = 1,000$ z_R instances is given in Table 4. The maximum absolute differences are plotted in Figure 4 for the *K*=1+2 partition which again provides the largest differences. Observed absolute differences by instances are larger than the differences observed for average z-values in Figure 3, as it should be expected, because of the stochastic nature of $(W)'s$ components in the 0-1 range. Looking at Figure 3 and at the statistical results in Table 4 shows that the bulk of points in the 1+2 partition lie in the 0 to 0.25 range; relatively fewer outliers with large trade-offs appear in the 0.3 to 0.55 range.

<u>Experiment 2</u>

K=6; the partition 6=2+2+2, e.g., (5,5,3,3,2,2), has been found to be representative enough for drawing conclusions, without having to compute all 416,675 different combinations. There are 6 such vectors with 3,375 merit-scores each, giving in total 20,250 different combinations, and more than 20 million random instances of the $W^{(R)}$ set to be computed with $N_R = 1,000$. As in experiment 1, Table 5 gives the statistics of the differences $\Delta_{59}, \Delta_{5\infty}$ with the reference set $W^{(5)}$ for all priority row vectors. Figure 5 plots the global scores z_5, z_9, z_∞. Again the maximum differences are observed when using the $W^{(\infty)}$ set, as to be expected, while they are quite small using the $W^{(9)}$ set.

Table 4. K=3. Statistics of $\max|z_5 - z_R|$ across $N_R = 1,000$ instances for all partitions. $\mu's$ are the means, $\sigma's$ the standard deviations, and *L's* the maximum absolute differences Δ.

Partitions of *K*=3	**#combinations**	**Representative priorities**	$\mu_{\Delta 5R}$	$\sigma_{\Delta 5R}$	$L_{\Delta 5R}$
1+1+1	750	(5,4,2)	0.13	0.07	0.43
1+2	300	(5,3,3)	0.13	0.11	0.53
2+1	300	(5,5,1)	0.07	0.05	0.23
3	35	(5,5,5)	0	0	0

Figure 4. K=3. Absolute maximum differences $\max|z_5 - z_R|$ across $N_R = 1,000$ instances for the 1+2 partition

Table 5. *K*=6. Statistics of the absolute differences $\Delta_{5*} = |z_5 - z_*|$ for the deterministic W sets for the 2+2+2 partition. $\mu's$ are the means, $\sigma's$ the standard deviations, *L's* the maximum values.

Partition of *K*=6	**#combinations**	**Representative priorities**	$\mu_{\Delta 59}$	$\sigma_{\Delta 59}$	$L_{\Delta 59}$	$\mu_{\Delta 54}$	$\sigma_{\Delta 54}$	$L_{\Delta 54}$
2+2+2	20,250	(5,5,3,3,2,2)	0.04	0.04	0.27	0.10	0.10	0.7

Figure 5. K=6. Comparison of the global scores for the 2+2+2 partition $z_5(\times)$, $z_9(*)$, $z_\infty(+)$. Every hundredth score is plotted

Figure 6. K=6. Comparison of the global scores for the 2+2+2 partition $z_5(\times)$, $z_R(o)$ for $N_R = 1,000$. Every hundredth score is plotted

Figure 7. Zoomed differences from Fig. 6 of $(z_5 - z_R)$

$(z_5 - z_R)$ are shown Figure 6 with $N_R = 1,000$ $\left(W^{(R)}\right)$ instances with hardly visible differences. The corresponding zoom is plotted in Fig. 7 showing again central symmetry, and similar conclusions are drawn as in experiment 1. The absolute differences between the unique z_5 and $N_R = 1,000$ z_R instances par combination are analysed in Table 6, and the maximum values per instance are displayed in Figure 8, for every hundredth vector combination. From this graph and statistical data it appears that, even though maximum differences up to about 0.45 appear within some instances, the bulk of values is below 0.20, meaning that using the $W^{(5)}$ set gives comparable global scores than $W^{(R)}$ in most situations.

Figure 8. K=6. $\max|z_5 - z_R|$ across $N_R = 1,000$ instances for the 2+2+2 partition. Every hundredth value is plotted

Table 6. K=6. Statistics of $\max|z_5 - z_R|$ across $N_R = 1,000$ instances for the 2+2+2 partition. μ's are the means, σ's the standard deviations, and *L's* the maximum values.

K=6 Partition	**#combinations**	**Representative priorities**	$\mu_{\Delta 5R}$	$\sigma_{\Delta 5R}$	$L_{\Delta 5R}$
(2,2,2)	20,250	(5,5,3,3,2,2)	0.10	0.06	0.43

The same type of conclusions is obtained as in the first experiment.

<u>Experiment 3</u>

K=5; all Pareto-optimal global scores for the merit row vector= $[5\ 4\ 3\ 2\ 2]$ are considered. The average score 3.2 is obtained in the case of complete indifference between the indicators, whatever the chosen approach with probability distribution is $z = 2(80\%) \quad z = 3(20\%)$. How does this plain result change when no knowledge on priorities is available, with other words, what are the Pareto-optimal global scores? Table 7 gives the global-score probability distributions for all 5^5 - 4^5=2,101 different (W) combinations, for the three sets $W^{(5)}, W^{(9)}, W^{(R)}$. Additional numerical experiments of the same kind - not reproduced here due to the limited space - confirm the validity of two general findings regarding global-score probability distributions:

a) All three sets give comparable global-score probability distributions for the more central global scores;
b) Larger spreads appear on the edges away from the central scores. This is easily understood because of the trade-off increase between extreme values from $W^{(5)}$ through $W^{(9)}$ to $W^{(R)}$; the latter set allows unlimited trade-offs, but with zero probability of becoming infinite, see the properties of Beta-distributions.

In conclusion for this third experiment the three $W^{(5)}, W^{(9)}, W^{(R)}$ sets bring similar global scores; still $W^{(9)}$ and $W^{(R)}$ scores come closer, because of the larger trade-offs towards the infinite edge.

Table 7. Global-score probability distribution P(z) of the merit row vector [5 4 3 2 2] obtained from $W^{(5)}, W^{(9)}, W^{(R)}$ sets

Priority Set	**P(z=1) %**	**P(z=2) %**	**P(z=3) %**	**P(z=4) %**	**P(z=5) %**	**Total**	$\bar{z}$
$\mathbf{W^{(5)}}$	0	3.5	73	23.5	0	100	3.2
$\mathbf{W^{(9)}}$	0	5,3	69.5	25.1	0.1	100	3.2
$\mathbf{W^{(R)}}$	0	5.0	70.3	24.4	0.3	100	3.2

4.3 Discussion

Although exhaustive comparisons are not feasible for all possible cases with different *K* and *R* values, it clearly results that all types of deterministic or stochastic (W) vectors analysed in three experiments give comparable global scores: this is expected to have general validity (a formal proof being out of reach, it is important to verify this conclusion by numerical comparisons in each situation). As verified in the 1st and 2nd experiment, the global scores obtained by Monte-Carlo simulations on $W^{(R)}$ sets marginally differ from $W^{(5)}$ results, even less from $W^{(9)}$, as confirmed in the third experiment. The global merit scores and their associated probabilities do therefore not change much with the choice of maximum trade-offs between indicators, as long as the former remain in a finite interval, say [5, 10], i.e., if the least important indicator represents at least 10 on a [1, 100] scale. More significant differences are observed when it is admitted that some indicators may be completely neglected, by choosing the set $W^{(\infty)} = \{4,3,2,1,0\}$. Looking at the result dispersion in Fig. 3, 4 and 7, 8, it is confirmed that the largest discrepancies stem from a limited number of outliers corresponding from extreme combinations, stemming from priority row vectors like (5,1,1) or (5,5,2,2,1,1), associated with merit row vectors (1,5,5) or (1,1,5,5,5,5). Such situations of large priorities ranges are unusual in practical problems: remember that in the most common case in which all indicators are indifferent – and thus the priority row vector is (5,5,…5), all weighed sums actually provide the same answer, which is just the average score. This is according to the plain averaging formula, traditionally used for rating in schools and universities. Except for situations implying large trade-offs, Monte-Carlo simulations are not indispensable, but they are still useful for giving the full range of trade-off possibilities. Such computing is fast, even for large instance numbers and *K* values, and certainly recommended.

5. Ranking a portfolio of policies

This section will be held to a minimum, as the main issue of the present article is first to present a sorting-out procedure of individual policies. When it comes to ranking policies within a portfolio, the normalised *(n×n) rank matrix* displays rank occupation probabilities:

$$\begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix} \qquad (29)$$

where p_{ij} is the probability for the *i-th* policy to occupy the *j-th* rank. Components in each row and column sum up to one. Assume the example in Table 8 displaying the global-score probability distribution of three policies; $\pm 5\%$ variants are indicated in parentheses for testing sensitivity.

Table 8. Probability distributions of global merit scores for three policies.

MG scores (%)	**1**	**2**	**3**	**4**	**5**
$\mathbf{P_1}$		45(40)	55(60)		
$\mathbf{P_2}$			62(57)	38(43)	
$\mathbf{P_3}$				80(85)	20(15)

It is easily verified that the probabilities R_{ij} of policy *i* occupying rank *j* are given by:

$$\begin{aligned} & R_{11} = \mu_{12} + \mu_{13}\mu_{24} + \mu_{13}\mu_{23}/2 \quad R_{12} = \mu_{13}\mu_{23}/2 \quad R_{13} = 0 \\ & R_{21} = \mu_{13}\mu_{23}/2 \quad R_{22} = \mu_{12}\mu_{23} + \mu_{13}\mu_{23}/2 + \mu_{24}\mu_{35} + \mu_{24}\mu_{34}/2 \\ & R_{23} = \mu_{24}\mu_{34}/2 \\ & R_{31} = 0 \quad R_{32} = \mu_{24}\mu_{34}/2 \quad R_{33} = \mu_{24}\mu_{34}/2 + \mu_{23} + \mu_{24}\mu_{35} \end{aligned} \qquad (30)$$

Just for easy understanding, remark that P_1 will be ranked first without competitor when its score is 2 or 3, while at the same time P_2 has score 3 or 4; when P_1 and P_2 share score 3, both occupy ranks 1 and 2 with half the probability each. P_1 can never have rank 3, and P_3 can never have rank 1. The rank matrix is Table 9 is obtained, with indicated in parentheses the $\pm 5\%$ changed values: the sensitivity of the rank probabilities on the merit score probabilities is rather small, and does not question the final ranking.

How to use the rank matrix to actually choose best policies will not be further discussed here. Though it is possible to bridge the policy sorting and ranking steps by means of analytical formulas, like (30), practitioners may prefer populating the rank matrix by much easier Monte-Carlo simulations on the global-score probability distributions of policies.

Table 9. The rank matrix obtained from the global scores in Table 8 with sensitivities.

Ranks Probabilities %	**1st**	**2nd**	**3rd**
R_1	83(83)	17(17)	0
R_2	17(17)	68(65)	15(18)
R_3		15(18)	85(82)

6. Conclusions and future research

The multiple-indicator procedure for sorting and ranking of policies presented in this article rests on the acceptance of the following three hypotheses:

- ✓ The first hypothesis is that presumably independent indicators for measuring performances are set up by policy-makers (PMs), none being negligible, and all having measurable objectives;
- ✓ The second hypothesis assumes that PMs are able to sort out indicators into priority classes regarding the importance of the attached objectives.
- ✓ The third hypothesis is that each proposed policy can be evaluated on a stand-alone basis by indicator, and globally for all of them, according to its merits in meeting the indicator objectives;

None of these hypotheses seems to be exaggeratedly demanding or restrictive for PMs. In the author's opinion the definition of indicators with measurable well-defined objectives is a prerequisite for successful decision processes, so that the sorting-out processes can take place efficiently. Because the human mind is limited in assigning objects to classes, the number of merit or priority classes is kept small, according to the 7 ± 2 golden rule. The author has chosen this number to be five, finer graining being possible, but perhaps not useful in many issues, depending also on the quality of available data. Once both sorting-out processes are completed, normalised scalar products, i.e., weighed sums, give sets of global merit scores and the corresponding probability distributions.

If one agrees that five equal intervals on a 0-100 scale are adequately representing priorities, values chosen from the set $\{5,4,3,2,1\}$ give a good evaluation basis. In the present article numerical experiments have shown the simplicity, fastness, and robustness of the approach avoiding all complications and interdependencies created by pairwise comparisons between policies (Kunsch, 2012). An accurate trade-off definition between priority classes is not critical. Because the approach is so simple and little computer-time consuming, many parametric and sensitivity studies are recommended in each practical situation for supporting the computation of global scores, i.e.:

- Changing the attribution to a merit or priority class in borderline cases;
- Changing the number of classes within the limits of the golden rule;

- Comparing results with different (W)-sets as described in the numerical experiments, etc.

Regarding future work, enumeration tools from group theory (Rotman, 1994) are to be developed for decreasing complexity, see section 4. Also the bridging of rating and ranking policies in a portfolio, only marginally addressed in section 5, would deserve additional development. Finally, some more research work could be done on how to define a set of pertinent and largely uncorrelated indicators, eliminating herewith redundancies.

References

Arnold, B.C., Balakrishnan, N. and Nagaraja, N. (1992) *A First course in Order Statistics*, New-York: John Wiley & Sons, Inc.

Bana e Costa, C.A., and Vansnick, J.C. (1994) 'MACBETH: An interactive path towards the construction of cardinal value functions', *International Transactions in Operational research*, Vol. 1, N°. 4, pp387-500.

Brans, J.P. and Mareschal, B. (2005) 'PROMETHEE methods', *In*: Figueira, J., Greco, S., Ehrgott, M. (eds.), *Multiple Criteria Decision Analysis: state of the art surveys*, (pp. 163-195), New York: Springer.

Buckley, J.J., Eslami, E., Feuring, Th. (2002) *Fuzzy Mathematics in Economics and Engineering*, Heidelberg:Phisica-Verlag, Springer.

Edwards, W., and Barron, F.H. (1994) 'SMART and SMARTER: Improved simple methods for multiattribute utility measurement', *Organizational Behavior and Human Decision Processes*, 60, pp. 306-325.

Figueira, J., Greco, S. and Ehrgott, M., eds., (2005) *Multiple Criteria Decision Analysis, State of the Art Surveys*. New York: Springer Science+Business Media, Inc.

Keeney, R., and Raiffa (1976) *Decision with Multiple Objectives: Preferences and value Tradeoffs.* Chichester: Wiley.

Kunsch, P.L. (2009) 'A Statistical Approach to Complex Multi-Criteria Decisions', *In:* Ruan, D., (ed.), *Computational Intelligence in Complex Decision Systems*, (Ch. 6), In Press, Paris: Atlantis Press.

Kunsch, P.L. (2010) 'A Statistical multi-criteria procedure with stochastic preferences', *International journal of MCDM*, Vol. 1/1, 2010, pp. 49-73.

Kunsch, P.L. (2012) 'Why pairwise comparison methods may fail in MCDM rankings', In: Klatte, D., Lüthi, H.J., Schmedders, K., (eds.), *Proceedings of OR 2011, Int. Conference on Operations Research, Zürich, Switzerland, August 30 to September 2*, Berlin:Springer, pp. 97-102.

Lahdelma, R., Hokkanen, J. and Salminen, P. (1998) 'SMAA – Stochastic multiobjective acceptability analysis', *European Journal of Operational Research*, Vol. 106, pp.137-143.

Miller, G.A. (1956) 'The magic number seven plus or minus two: some limits on our capacity for processing information', *Psychological review*, Vol. 13, pp.81-97.

Rotman, Joseph (1994), *An introduction to the theory of groups*, New York: Springer-Verlag.

Roy, B. (1996) *Multicriteria Methodology for Decision Aiding*, Dordrecht: Kluwer.

Saaty, T.L. (1980) The Analytic Hierarchy Process. New YorkMcGraw-Hill.

Vincke, Ph. (1992) *Multi-criteria decision aid*, Chichester: John Wiley&Sons.

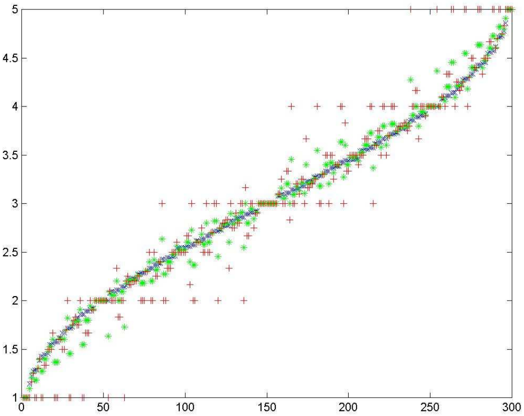

Figure 1. K=3. Comparison of the global scores for the 1+2 partition $z_5(\times)$, $z_9(*)$, $z_\infty(+)$

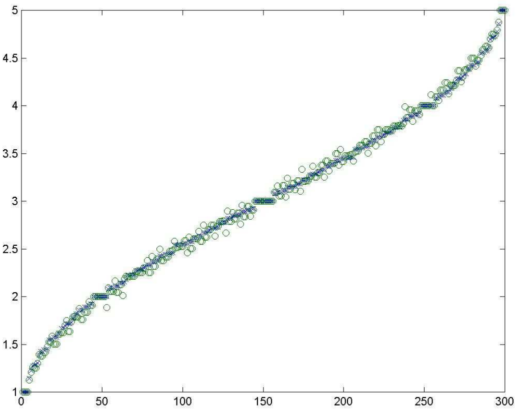

Figure 2. K=3. Comparison of the global scores for the 1+2 partition $z_5(\times)$, $z_R(o)$ for $N_R = 1{,}000$

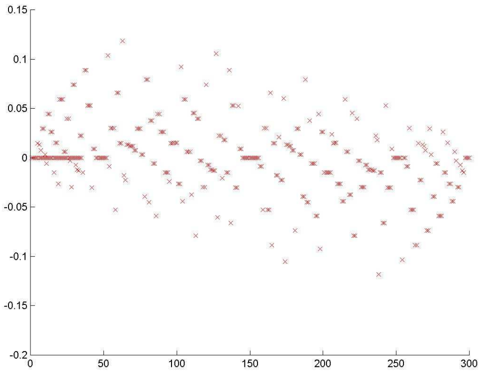

Figure 3. Zoomed differences from Fig. 2 of $\left(z_5 - z_R\right)$

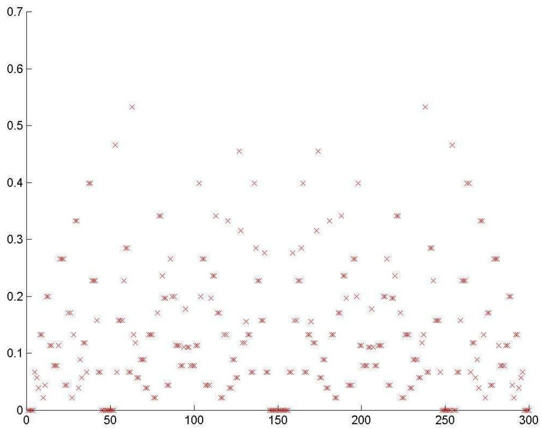

Figure 4. K=3. Absolute maximum differences $\max|z_5 - z_R|$ across $N_R = 1{,}000$ instances for the 1+2 partition

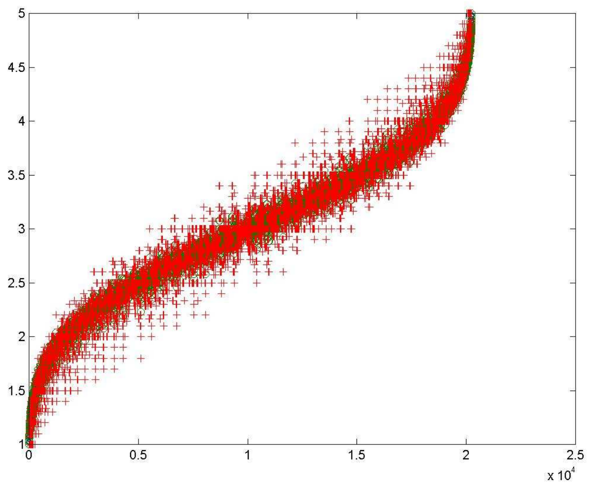

Figure 5. K=6. Comparison of the global scores for the 2+2+2 partition $z_5(\times)$, $z_9(*)$, $z_\infty(+)$. Every hundredth score is plotted

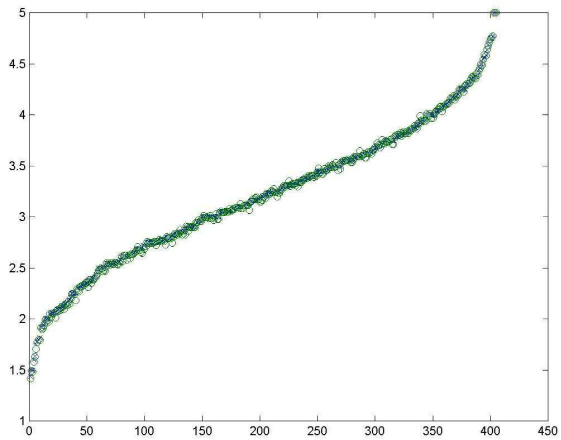

Figure 6. K=6. Comparison of the global scores for the 2+2+2 partition $z_5(\times)$, $z_R(o)$ for $N_R = 1{,}000$. Every hundredth score is plotted

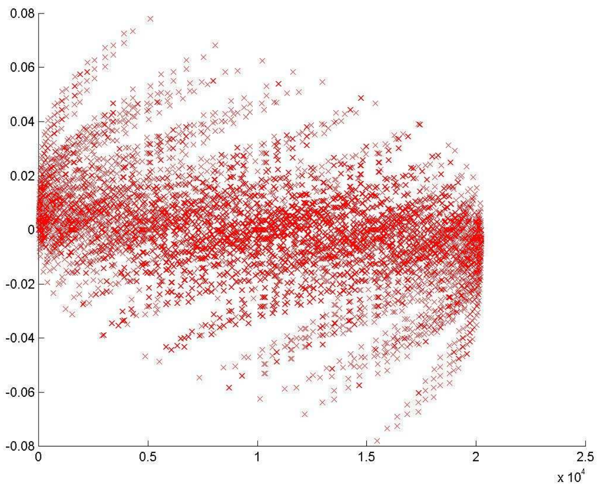

Figure 7. Zoomed differences from Fig. 6 of $\left(z_5 - z_R\right)$

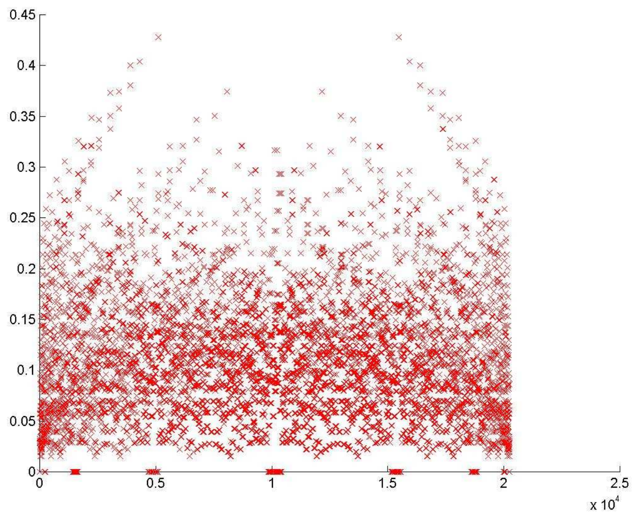

Figure 8. K=6. $\max | z_5 - z_R |$ across $N_R = 1,000$ instances for the 2+2+2 partition. Every hundredth value is plotted